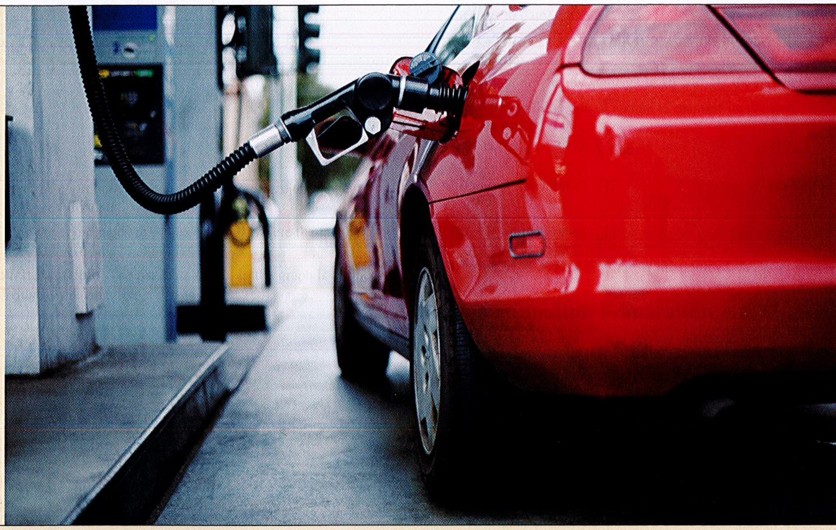
CHAPTER

5

Applications of Derivatives



An automobile's gas mileage is a function of many variables, including road surface, tire type, velocity, fuel octane rating, road angle, and the speed and direction of the wind. If we look only at velocity's effect on gas mileage, the mileage of a certain car can be approximated by

$$m(v) = 0.00015v^3 - 0.032v^2 + 1.8v + 1.7$$

(where v is velocity).

At what speed should you drive this car to obtain the best gas mileage? The ideas in Section 5.1 will help you find the answer.

CHAPTER 5 Overview

In the past, when virtually all graphing was done by hand—often laboriously—derivatives were the key tool used to sketch the graph of a function. Now we can graph a function quickly, and usually correctly, using a grapher. However, confirmation of much of what we see and conclude true from a grapher view must still come from calculus.

This chapter shows how to draw conclusions from derivatives about the extreme values of a function and about the general shape of a function's graph. We will also see how a tangent line captures the shape of a curve near the point of tangency, how to deduce rates of change we cannot measure from rates of change we already know, and how to find a function when we know only its first derivative and its value at a single point. The key to recovering functions from derivatives is the Mean Value Theorem, a theorem whose corollaries provide the gateway to *integral calculus,* which we begin in Chapter 6.

5.1 Extreme Values of Functions

What you will learn about . . .

- Absolute (Global) Extreme Values
- Local (Relative) Extreme Values
- Finding Extreme Values

and why . . .

Finding maximum and minimum values of functions, called optimization, is an important issue in real-world problems.

Absolute (Global) Extreme Values

One of the most useful things we can learn from a function's derivative is whether the function assumes any maximum or minimum values on a given interval and where these values are located if it does. Once we know how to find a function's extreme values, we will be able to answer such questions as "What is the most effective size for a dose of medicine?" and "What is the least expensive way to pipe oil from an offshore well to a refinery down the coast?" We will see how to answer questions like these in Section 5.4.

DEFINITION Absolute Extreme Values

Let f be a function with domain D. Then $f(c)$ is the

(a) **absolute maximum value** on D if and only if $f(x) \leq f(c)$ for all x in D.
(b) **absolute minimum value** on D if and only if $f(x) \geq f(c)$ for all x in D.

Absolute (or **global**) maximum and minimum values are also called **absolute extrema** (plural of the Latin *extremum*). We often omit the term "absolute" or "global" and just say maximum and minimum.

Example 1 shows that extreme values can occur at interior points or endpoints of intervals.

EXAMPLE 1 Exploring Extreme Values

On $[-\pi/2, \pi/2]$, $f(x) = \cos x$ takes on a maximum value of 1 (once) and a minimum value of 0 (twice). The function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 5.1). *Now Try Exercise 1.*

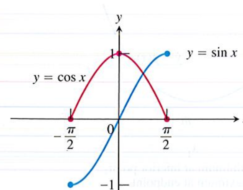


Figure 5.1 (Example 1)

Functions with the same defining rule can have different extrema, depending on the domain.

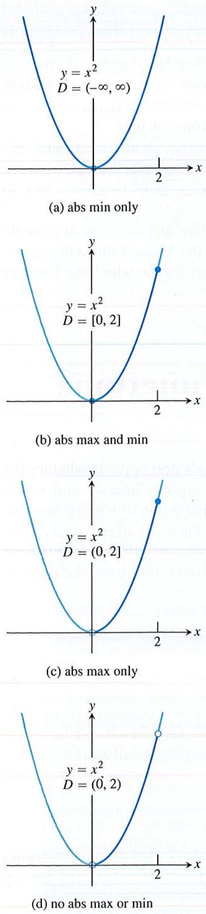


Figure 5.2 (Example 2)

EXAMPLE 2 Exploring Absolute Extrema

The absolute extrema of the following functions on their domains can be seen in Figure 5.2.

	Function Rule	Domain D	Absolute Extrema on D
(a)	$y = x^2$	$(-\infty, \infty)$	No absolute maximum Absolute minimum of 0 at $x = 0$
(b)	$y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$ Absolute minimum of 0 at $x = 0$
(c)	$y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$ No absolute minimum
(d)	$y = x^2$	$(0, 2)$	No absolute extrema

Now Try Exercise 3.

Example 2 shows that a function may fail to have a maximum or minimum value. This cannot happen with a continuous function on a finite closed interval.

THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a maximum value and a minimum value on the interval. (Figure 5.3)

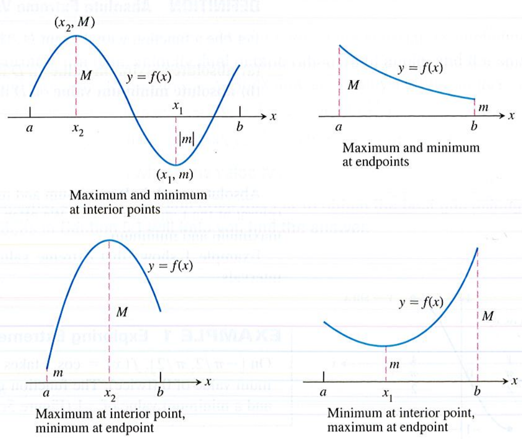


Figure 5.3 Some possibilities for a continuous function's maximum (M) and minimum (m) on a closed interval $[a, b]$.

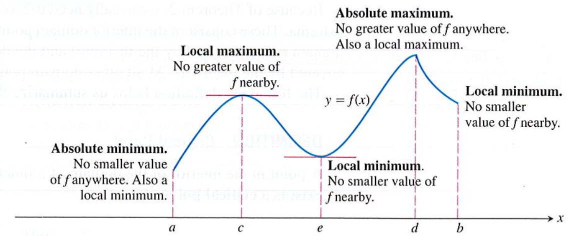


Figure 5.4 Classifying extreme values.

Local (Relative) Extreme Values

Figure 5.4 shows a graph with five points where a function has extreme values on its domain $[a, b]$. The function's absolute minimum occurs at a even though at e the function's value is smaller than at any other point *nearby*. The curve rises to the left and falls to the right around c, making $f(c)$ a maximum locally. The function attains its absolute maximum at d.

DEFINITION Local Extreme Values

Let c be an interior point of the domain of the function f. Then $f(c)$ is a

(a) **local maximum value** at c if and only if $f(x) \leq f(c)$ for all x in some open interval containing c.

(b) **local minimum value** at c if and only if $f(x) \geq f(c)$ for all x in some open interval containing c.

A function f has a local maximum or local minimum *at an endpoint* c if the appropriate inequality holds for all x in some half-open domain interval containing c.

Local extrema are also called **relative extrema.**

An **absolute extremum** is also a local extremum, because being an extreme value overall makes it an extreme value in its immediate neighborhood. Hence, *a list of local extrema will automatically include absolute extrema if there are any.*

Finding Extreme Values

The interior domain points where the function in Figure 5.4 has local extreme values are points where either f' is zero or f' does not exist. This is generally the case, as we see from the following theorem.

THEOREM 2 Local Extreme Values

If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then

$$f'(c) = 0.$$

Because of Theorem 2, we usually need to look at only a few points to find a function's extrema. These consist of the interior domain points where $f' = 0$ or f' does not exist (the domain points covered by the theorem) and the domain endpoints (the domain points not covered by the theorem). At all other domain points, $f' > 0$ or $f' < 0$.

The following definition helps us summarize these findings.

DEFINITION Critical Point

A point in the interior of the domain of a function f at which $f' = 0$ or f' does not exist is a **critical point** of f.

Thus, in summary, extreme values occur only at critical points and endpoints.

DEFINITION Stationary Point

A point in the interior of the domain of a function f at which $f' = 0$ is called a *stationary point* of f.

A stationary point can be a minimum, a maximum, or an inflection point. Note that critical points and stationary points of a function f are *not necessarily* the same. See Examples 3 and 5.

EXAMPLE 3 Finding Absolute Extrema

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

SOLUTION

Solve Analytically We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

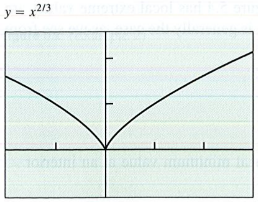
has no zeros but is undefined at $x = 0$. The values of f at this one critical point and at the endpoints are

$$\begin{aligned} &\text{Critical point value:} && f(0) = 0; \\ &\text{Endpoint values:} && f(-2) = (-2)^{2/3} = \sqrt[3]{4}; \\ &&& f(3) = (3)^{2/3} = \sqrt[3]{9}. \end{aligned}$$

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and occurs at the interior point $x = 0$.

Support Graphically The graph in Figure 5.5 suggests that f has an absolute maximum value of about 2 at $x = 3$ and an absolute minimum value of 0 at $x = 0$. The critical point $(0, 0)$ is **not** a stationary point. ***Now Try Exercise 11.***

$y = x^{2/3}$



$[-2, 3]$ by $[-1, 2.5]$

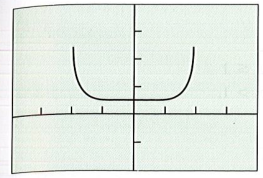
Figure 5.5 (Example 3)

In Example 4, we investigate the reciprocal of the function whose graph was drawn in Example 3 of Section 1.2 to illustrate "grapher failure."

EXAMPLE 4 Finding Extreme Values

Find the extreme values of $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$.

continued



$[-4, 4]$ by $[-2, 4]$

Figure 5.6 The graph of

$$f(x) = \frac{1}{\sqrt{4 - x^2}}.$$

(Example 4)

SOLUTION

Solve Graphically Figure 5.6 suggests that f has an absolute minimum of about 0.5 at $x = 0$. There also appear to be local maxima at $x = -2$ and $x = 2$. However, f is not defined at these points and there do not appear to be maxima anywhere else.

Confirm Analytically The function f is defined only for $4 - x^2 > 0$, so its domain is the open interval $(-2, 2)$. The domain has no endpoints, so all the extreme values must occur at critical points. We rewrite the formula for f to find f':

$$f(x) = \frac{1}{\sqrt{4 - x^2}} = (4 - x^2)^{-1/2}$$

Thus,

$$f'(x) = -\frac{1}{2}(4 - x^2)^{-3/2}(-2x) = \frac{x}{(4 - x^2)^{3/2}}.$$

The only critical point in the domain $(-2, 2)$ is $x = 0$. The value

$$f(0) = \frac{1}{\sqrt{4 - 0^2}} = \frac{1}{2}$$

is therefore the sole candidate for an extreme value.

To determine whether $1/2$ is an extreme value of f, we examine the formula

$$f(x) = \frac{1}{\sqrt{4 - x^2}}.$$

As x moves away from 0 on either side, the denominator gets smaller, the values of f increase, and the graph rises. We have a minimum value at $x = 0$, and the minimum is absolute.

The function has no maxima, either local or absolute. This does not violate Theorem 1 (The Extreme Value Theorem) because here f is defined on an *open* interval. To invoke Theorem 1's guarantee of extreme points, the interval must be closed. ***Now Try Exercise 25.***

While a function's extrema can occur only at critical points and endpoints, not every critical point or endpoint signals the presence of an extreme value. Figure 5.7 illustrates this for interior points. Exercise 55 describes a function that fails to assume an extreme value at an endpoint of its domain.

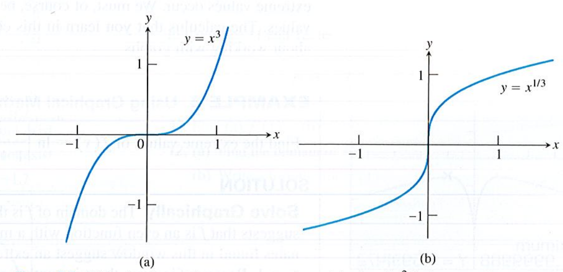


Figure 5.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

EXAMPLE 5 Finding Extreme Values

Find the extreme values of

$$f(x) = \begin{cases} 5 - 2x^2, & x \le 1 \\ x + 2, & x > 1. \end{cases}$$

SOLUTION

Solve Analytically For $x \ne 1$, the derivative is

$$f'(x) = \begin{cases} \dfrac{d}{dx}(5 - 2x^2) = -4x, & x < 1 \\ \dfrac{d}{dx}(x + 2) = 1, & x > 1. \end{cases}$$

The only point where $f' = 0$ is $x = 0$. What happens at $x = 1$?

At $x = 1$, the right- and left-hand derivatives are, respectively,

$$\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1 + h) + 2 - 3}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1.$$

$$\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^-} \frac{5 - 2(1 + h)^2 - 3}{h}$$

$$= \lim_{h \to 0^-} \frac{-2h(2 + h)}{h} = -4.$$

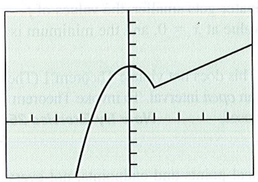
Since these one-sided derivatives differ, f has no derivative at $x = 1$, and 1 is a second critical point of f.

The domain $(-\infty, \infty)$ has no endpoints, so the only values of f that might be local extrema are those at the critical points:

$$f(0) = 5 \quad \text{and} \quad f(1) = 3$$

From the formula for f, we see that the values of f immediately to either side of $x = 0$ are less than 5, so 5 is a local maximum. Similarly, the values of f immediately to either side of $x = 1$ are greater than 3, so 3 is a local minimum.

Support Graphically The graph in Figure 5.8 suggests that $f'(0) = 0$ and that $f'(1)$ does not exist. There appears to be a local maximum value of 5 at $x = 0$ and a local minimum value of 3 at $x = 1$. The point $(0, 50)$ is the only stationary point. *Now Try Exercise 41.*



[–5, 5] by [–5, 10]

Figure 5.8 The function in Example 5.

Most graphing calculators have built-in methods to find the coordinates of points where extreme values occur. We must, of course, be sure that we use correct graphs to find these values. The calculus that you learn in this chapter should make you feel more confident about working with graphs.

EXAMPLE 6 Using Graphical Methods

Find the extreme values of $f(x) = \ln \left| \dfrac{x}{1 + x^2} \right|$.

SOLUTION

Solve Graphically The domain of f is the set of all nonzero real numbers. Figure 5.9 suggests that f is an even function with a maximum value at two points. The coordinates found in this window suggest an extreme value of about -0.69 at approximately $x = 1$. Because f is even, there is another extreme of the same value at approximately $x = -1$. The figure also suggests a minimum value at $x = 0$, but f is not defined there.

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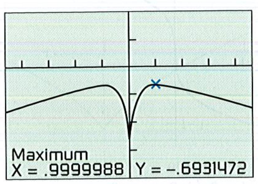


[–4.5, 4.5] by [–4, 2]

Figure 5.9 The function in Example 6.

Confirm Analytically The derivative

$$f'(x) = \frac{1 - x^2}{x(1 + x^2)}$$

is defined at every point of the function's domain. The critical points where $f'(x) = 0$ are $x = 1$ and $x = -1$. The corresponding values of f are both $\ln(1/2) = -\ln 2 \approx -0.69$. *Now Try Exercise 37.*

EXPLORATION 1 Finding Extreme Values

Let $f(x) = \left| \dfrac{x}{x^2 + 1} \right|, -2 \le x \le 2$.

1. Determine graphically the extreme values of f and where they occur. Find f' at these values of x.
2. Graph f and f' (or NDER $(f(x), x, x)$) in the same viewing window. Comment on the relationship between the graphs.
3. Find a formula for $f'(x)$.

Quick Review 5.1 *(For help, go to Sections 1.2, 2.1, 3.5, and 4.1.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–4, find the first derivative of the function.

1. $f(x) = \sqrt{4 - x}$
2. $f(x) = \dfrac{2}{\sqrt{9 - x^2}}$
3. $g(x) = \cos(\ln x)$
4. $h(x) = e^{2x}$

In Exercises 5–8, match the table with a graph of $f(x)$.

5.

x	$f'(x)$
a	0
b	0
c	5

6.

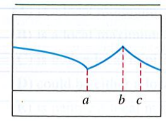
x	$f'(x)$
a	0
b	0
c	-5

7.

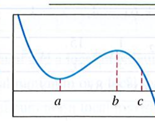
x	$f'(x)$
a	does not exist
b	0
c	-2

8.

x	$f'(x)$
a	does not exist
b	does not exist
c	-1.7



(a)



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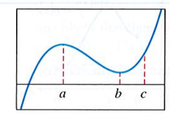
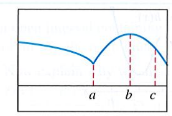


(c)



(d)

In Exercises 9 and 10, find the limit for

$$f(x) = \frac{2}{\sqrt{9 - x^2}}.$$

9. $\displaystyle\lim_{x \to 3^-} f(x)$

10. $\displaystyle\lim_{x \to -3^+} f(x)$

In Exercises 11 and 12, let

$$f(x) = \begin{cases} x^3 - 2x, & x \le 2 \\ x + 2, & x > 2. \end{cases}$$

11. Find **(a)** $f'(1)$, **(b)** $f'(3)$, **(c)** $f'(2)$.

12. **(a)** Find the domain of f'.

(b) Write a formula for $f'(x)$.

Section 5.1 Exercises

In Exercises 1–4, find the extreme values and where they occur.

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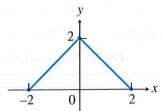

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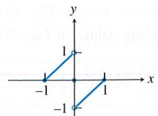

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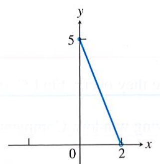

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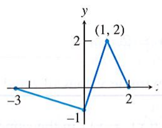

In Exercises 5–10, identify each x-value at which any absolute extreme value occurs. Explain how your answer is consistent with the Extreme Value Theorem.

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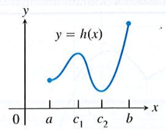

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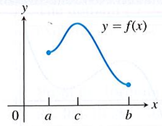

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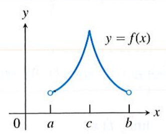

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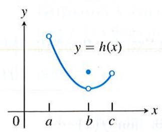

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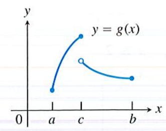

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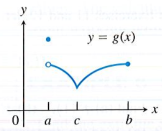

In Exercises 11–18, use analytic methods to find the extreme values of the function on the interval and where they occur. Identify any critical points that are *not* stationary points.

11. $f(x) = \dfrac{1}{x} + \ln x, \quad 0.5 \le x \le 4$

12. $g(x) = e^{-x}, \quad -1 \le x \le 1$

13. $h(x) = \ln(x + 1), \quad 0 \le x \le 3$

14. $k(x) = e^{-x^2}, \quad -\infty < x < \infty$

15. $f(x) = \sin\left(x + \dfrac{\pi}{4}\right), \quad 0 \le x \le \dfrac{7\pi}{4}$

16. $g(x) = \sec x, \quad -\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$

17. $f(x) = x^{2/5}, \quad -3 \le x < 1$

18. $f(x) = x^{3/5}, \quad -2 < x \le 3$

In Exercises 19–30, find the extreme values of the function and where they occur.

19. $y = 2x^2 - 8x + 9$

20. $y = x^3 - 2x + 4$

21. $y = x^3 + x^2 - 8x + 5$

22. $y = x^3 - 3x^2 + 3x - 2$

23. $y = \sqrt{x^2 - 1}$

24. $y = \dfrac{1}{x^2 - 1}$

25. $y = \dfrac{1}{\sqrt{1 - x^2}}$

26. $y = \dfrac{1}{\sqrt[3]{1 - x^2}}$

27. $y = \sqrt{3 + 2x - x^2}$

28. $y = \dfrac{3}{2}x^4 + 4x^3 - 9x^2 + 10$

29. $y = \dfrac{x}{x^2 + 1}$

30. $y = \dfrac{x + 1}{x^2 + 2x + 2}$

Group Activity In Exercises 31–34, find the extreme values of the function on the interval and where they occur.

31. $f(x) = |x - 2| + |x + 3|, \quad -5 \le x \le 5$

32. $g(x) = |x - 1| - |x - 5|, \quad -2 \le x \le 7$

33. $h(x) = |x + 2| - |x - 3|, \quad -\infty < x < \infty$

34. $k(x) = |x + 1| + |x - 3|, \quad -\infty < x < \infty$

In Exercises 35–42, identify the critical points and determine the local extreme values. Identify which critical points are *not* stationary points.

35. $y = x^{2/3}(x + 2)$

36. $y = x^{2/3}(x^2 - 4)$

37. $y = x\sqrt{4 - x^2}$

38. $y = x^2\sqrt{3 - x}$

39. $y = \begin{cases} 4 - 2x, & x \le 1 \\ x + 1, & x > 1 \end{cases}$

40. $y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \ge 0 \end{cases}$

41. $y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$

42. $y = \begin{cases} -\dfrac{1}{4}x^2 - \dfrac{1}{2}x + \dfrac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

43. Writing to Learn The function

$$V(x) = x(10 - 2x)(16 - 2x), \quad 0 < x < 5.$$

models the volume of a box.

(a) Find the extreme values of V.

(b) Interpret any values found in (a) in terms of volume of the box.

(c) Support your analytic answer to part (a) graphically.

44. Writing to Learn The function

$$P(x) = 2x + \frac{200}{x}, \quad 0 < x < \infty,$$

models the perimeter of a rectangle of dimensions x by $100/x$.

(a) Find any extreme values of P.

(b) Give an interpretation in terms of perimeter of the rectangle for any values found in (a).

Standardized Test Questions

45. True or False If $f(c)$ is a local maximum of a continuous function f on an open interval (a, b), then $f'(c) = 0$. Justify your answer.

46. True or False If m is a local minimum and M is a local maximum of a continuous function f on (a, b), then $m < M$. Justify your answer.

47. Multiple Choice Which of the following values is the absolute maximum of the function $f(x) = 4x - x^2 + 6$ on the interval $[0, 4]$?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 10

48. Multiple Choice If f is a continuous, decreasing function on $[0, 10]$ with a critical point at $(4, 2)$, which of the following statements *must be false*?

(A) $f(10)$ is an absolute minimum of f on $[0, 10]$.

(B) $f(4)$ is neither a relative maximum nor a relative minimum.

(C) $f'(4)$ does not exist.

(D) $f'(4) = 0$

(E) $f'(4) < 0$

49. Multiple Choice Which of the following functions has exactly two local extrema on its domain?

(A) $f(x) = |x - 2|$

(B) $f(x) = x^3 - 6x + 5$

(C) $f(x) = x^3 + 6x - 5$

(D) $f(x) = \tan x$

(E) $f(x) = x + \ln x$

50. Multiple Choice If an even function f with domain all real numbers has a local maximum at $x = a$, then $f(-a)$

(A) is a local minimum.

(B) is a local maximum.

(C) is both a local minimum and a local maximum.

(D) could be either a local minimum or a local maximum.

(E) is neither a local minimum nor a local maximum.

Explorations

In Exercises 51 and 52, give reasons for your answers.

51. Writing to Learn Let $f(x) = (x - 2)^{2/3}$.

(a) Does $f'(2)$ exist?

(b) Show that the only local extreme value of f occurs at $x = 2$.

(c) Does the result in (b) contradict the Extreme Value Theorem?

(d) Repeat parts (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2 by a.

52. Writing to Learn Let $f(x) = |x^3 - 9x|$.

(a) Does $f'(0)$ exist?

(b) Does $f'(3)$ exist?

(c) Does $f'(-3)$ exist?

(d) Determine all extrema of f.

Extending the Ideas

53. *Cubic Functions* Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$

(a) Show that f can have 0, 1, or 2 critical points. Give examples and graphs to support your argument.

(b) How many local extreme values can f have?

54. *Proving Theorem 2* Assume that the function f has a local maximum value at the interior point c of its domain and that $f'(c)$ exists.

(a) Show that there is an open interval containing c such that $f(x) - f(c) \le 0$ for all x in the open interval.

(b) **Writing to Learn** Now explain why we may say

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0.$$

(c) **Writing to Learn** Now explain why we may say

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge 0.$$

(d) **Writing to Learn** Explain how parts (b) and (c) allow us to conclude $f'(c) = 0$.

(e) **Writing to Learn** Give a similar argument if f has a local minimum value at an interior point.

55. *Functions with No Extreme Values at Endpoints*

(a) Graph the function

$$f(x) = \begin{cases} \sin\dfrac{1}{x}, & x > 0 \\ 0, & x = 0. \end{cases}$$

Explain why $f(0) = 0$ is not a local extreme value of f.

(b) **Group Activity** Construct a function of your own that fails to have an extreme value at a domain endpoint.